Classification and Statistics of Cut-and-Project Sets

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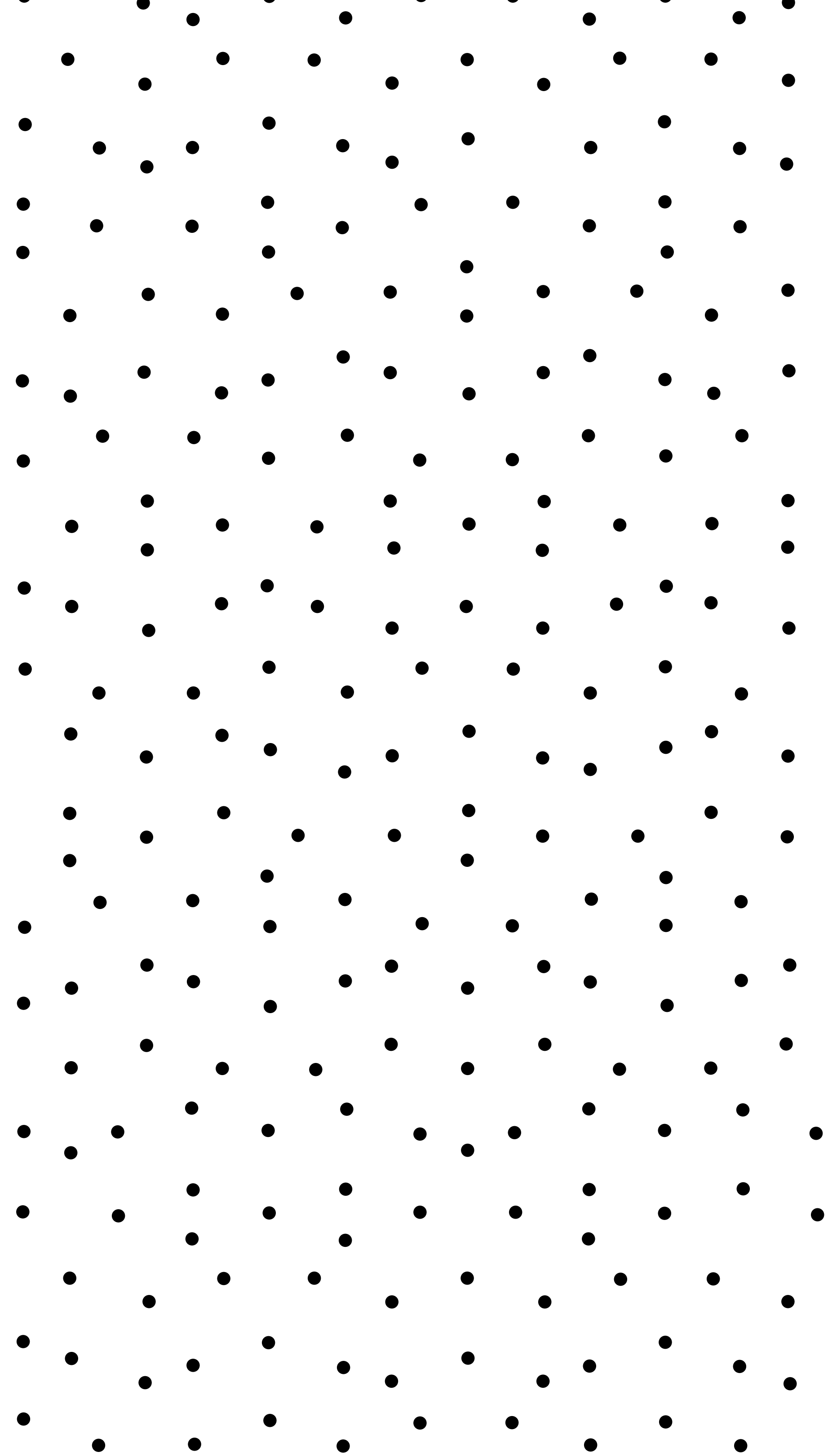
HUJI Dynamics Seminar, 2023

Joint with René Rühr and Barak Weiss

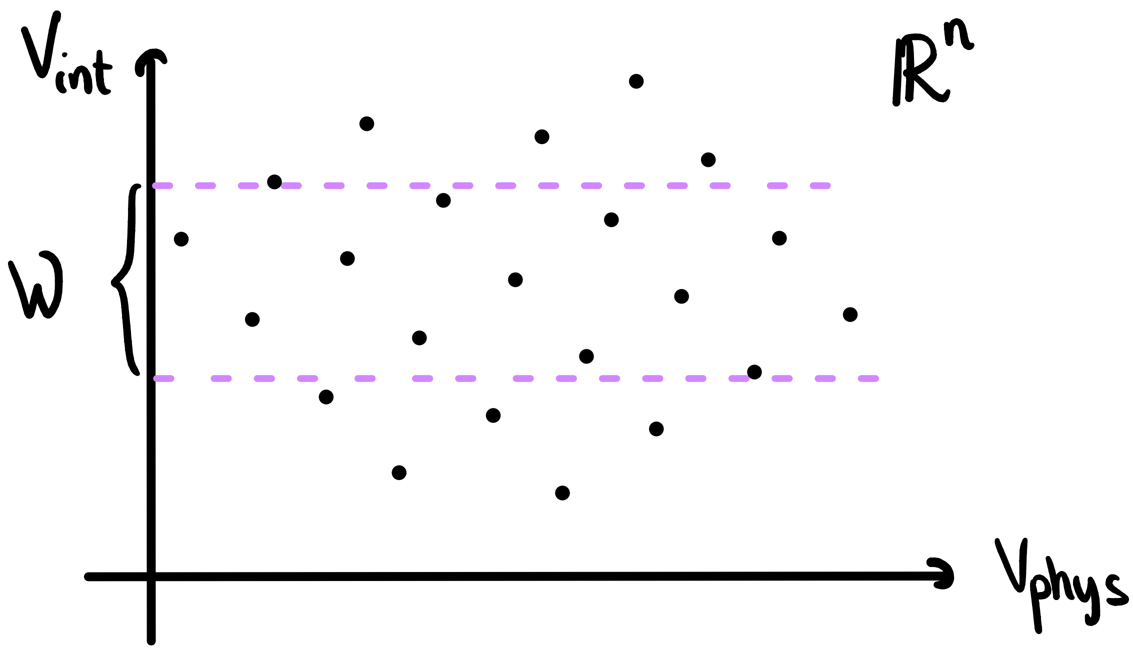
Plan of Talk

- Cut-and-project sets
- "Spaces of quasicrystals" and classification of measures
- Counting points and patches



Cut-and-Project Sets

- Fix a decomposition $\mathbb{R}^n = \underbrace{\mathbb{R}^d}_{\text{physical space } V_{phys}} \oplus \underbrace{\mathbb{R}^m}_{\text{internal space } V_{int}}$ with projections π_{phys} and π_{int}
- Fix a lattice or a grid $\mathcal{L} \subset \mathbb{R}^n$ and a window $W \subset V_{int}$



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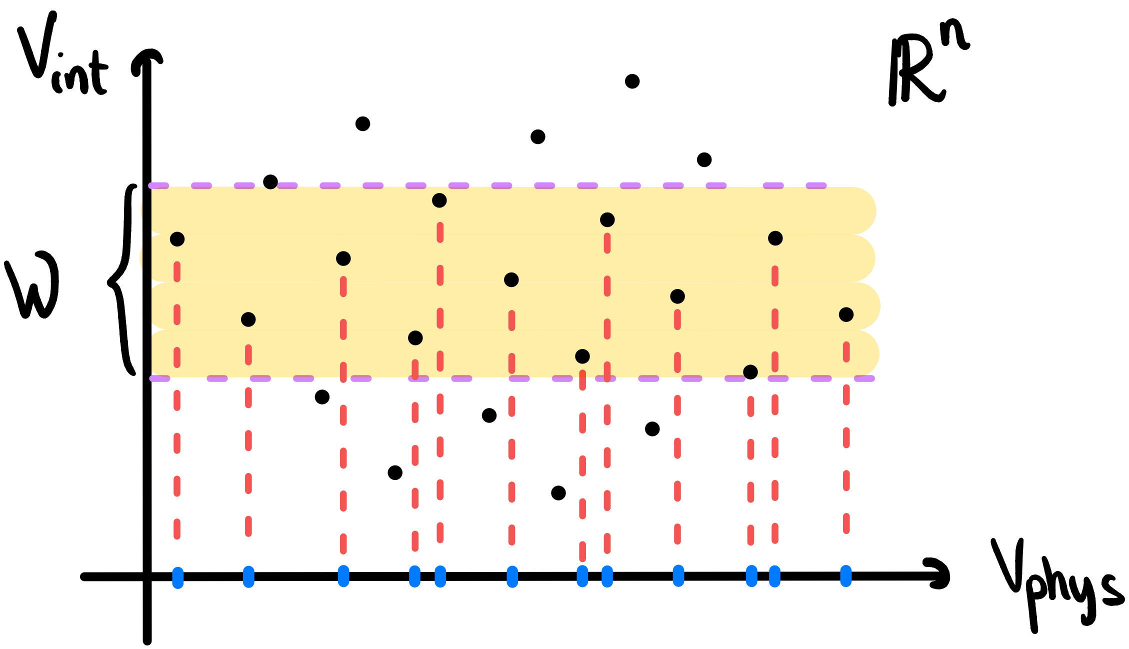
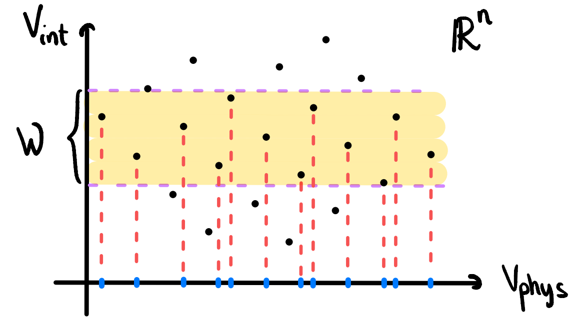


$$\Lambda = \Lambda(\mathcal{L}, W) := \pi_{phys}\left(\mathcal{L} \cap \pi_{int}^{-1}(W)\right) \subset V_{phys}$$

The cut-and-project set associated with $\mathcal{L}$ and W

Assumptions and Basic Properties

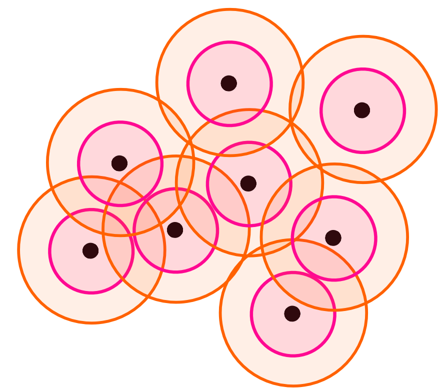


$\Lambda(\mathcal{L}, W)$ is irreducible if $\overline{\pi_{int}(\mathcal{L})} = V_{int}$,

π_{phys} is 1-1 on $\mathcal{L}$, and if W is regular:

- Bounded $\Rightarrow$ Λ is uniformly discrete
- Non-empty interior $\Rightarrow$ Λ is relatively dense

$\Rightarrow$ Delone



- Boundary of measure zero $\Rightarrow$ Λ has asymptotic density

$$D(\Lambda) := \lim_{T \to \infty} \frac{\#\{\Lambda \cap B(0,T)\}}{\mathrm{vol}(B(0,T))} = \frac{m(W)}{\mathrm{covol}(\mathcal{L})}$$

For $x \in \pi_{phys}(\mathcal{L})$ define $x^* := \pi_{int}(\pi_{phys}^{-1}(x))$

If $*$ is 1-1 $\Rightarrow$ Λ has no periods

Motivations and Relations

- Geometric

A Delone set Γ is Meyer if $\Gamma - \Gamma$ is also Delone

$\Rightarrow$ Every $\Lambda(\mathcal{L}, W)$ is Meyer

Meyer Every Meyer set is contained in some $\Lambda(\mathcal{L}, W)$

- Dynamical

Delone sets are elements of $\mathcal{C}(\mathbb{R}^d) :=$ space of closed subsets of $\mathbb{R}^d$ which carries a natural topology.

Set $X_\Lambda := \overline{\{\Lambda - t \mid t \in \mathbb{R}^d\}}$, then

Hof, Schlottmann Dynamics of $(X_\Lambda, \mathbb{R}^d) \Rightarrow$ pure point diffraction

Motivations and Relations

- **Arithmetic**

Interesting sets have representations as $\Lambda(\mathcal{L}, W)$:

I An algebraic integer > 1 is Pisot if all its conjugates lie inside the unit disk.

Let $K = \mathbb{Q}(\sqrt{2})$ with a ring of integers $\mathcal{O}_K$, and set $\mathcal{L} = \{(x, \bar{x}) \mid x \in \mathcal{O}_K\}$ the Minkowski embedding

$\Rightarrow$ Pisot numbers in $\mathcal{O}_K = \Lambda(\mathcal{L}, (-1,1)) \cap (1, \infty)$

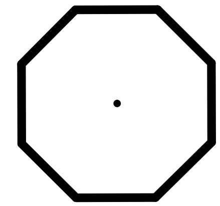
II Relaxing assumption and allowing V_{int} = adeles

$\Rightarrow$ primitive vectors

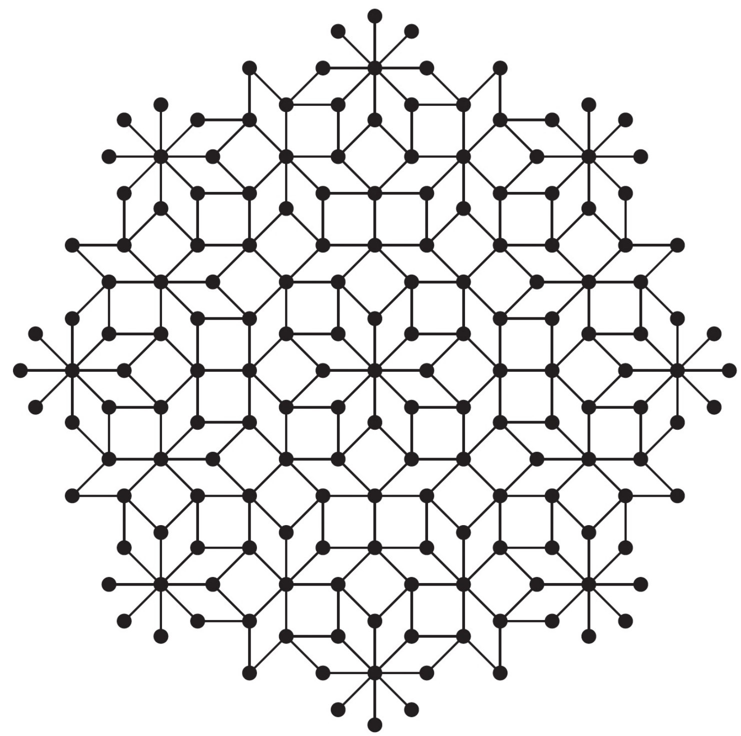
Example: The Ammann-Beenker Point Set

Let $K = \mathbb{Q}(\sqrt{2})$, and set

$$\mathcal{L} = \{(x_1, x_2, \overline{x}_1, \overline{x}_2) \mid x_1, x_2 \in \mathcal{O}_K,\ \tfrac{1}{\sqrt{2}}(x_1 - x_2) \in \mathcal{O}_K\}$$

$W =$  $\}\sqrt{2}$

$\Lambda(\mathcal{L}, W)$ is then the vertex set of the Ammann-Beenker tiling, which can also be defined via a substitution rule with inflation constant $\lambda = 1 + \sqrt{2}$



From Baake and Grimm's Aperiodic Order Vol 1

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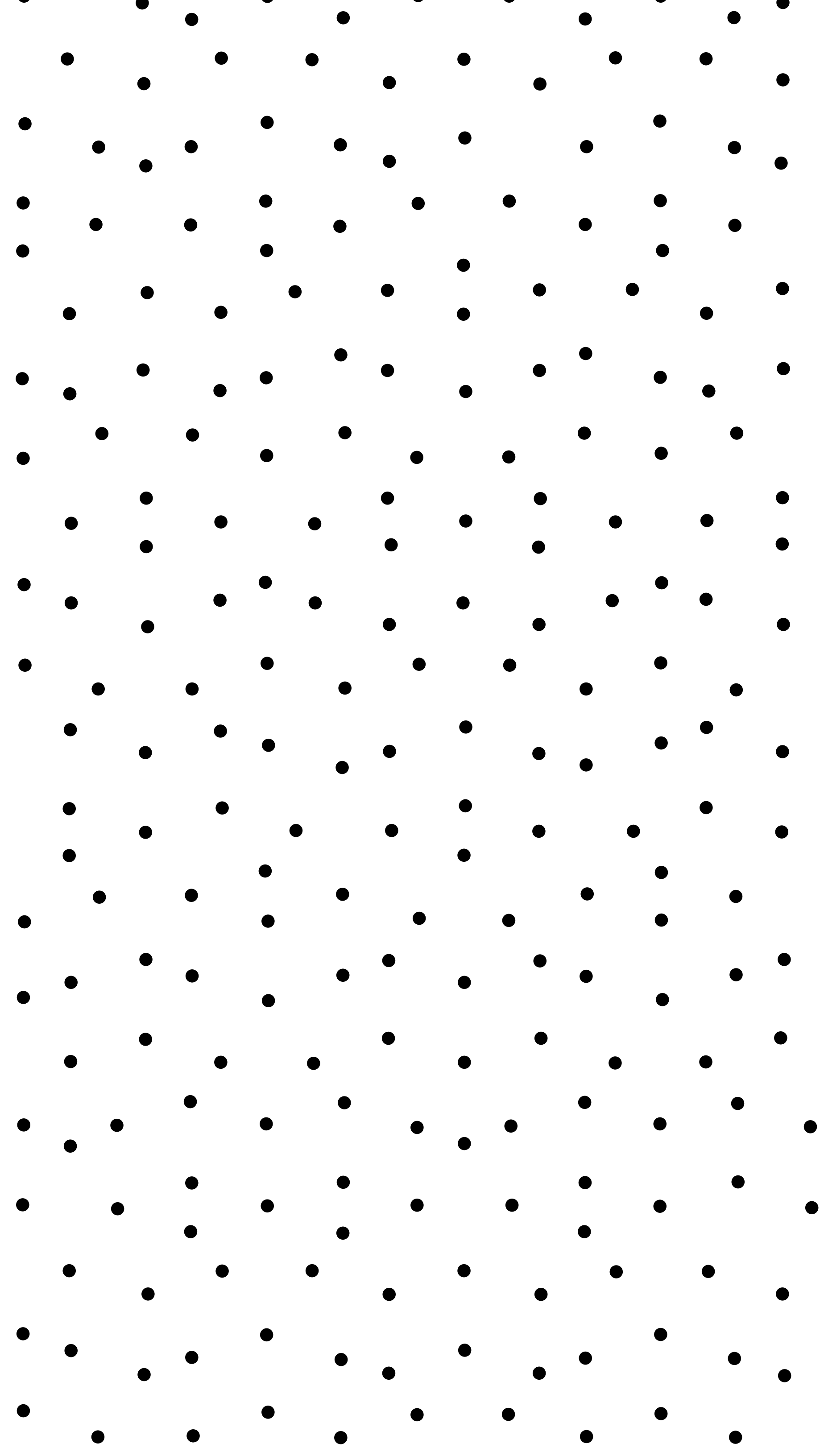

Action of $ASL_d(\mathbb{R})$ and Main Goals

$$ASL_d(\mathbb{R}) = SL_d(\mathbb{R}) \ltimes \mathbb{R}^d = \left\{ \begin{array}{c} \text{volume and orientation} \\ \text{preserving affine maps } \mathbb{R}^d \to \mathbb{R}^d \end{array} \right\}$$

- Describe probability measures on cut-and-project sets in $\mathbb{R}^d$, $d \geq 2$, that are affine group invariant and ergodic.

  $\Rightarrow$ We obtain a complete classification

- Describe counting statistics for typical cut-and-project sets with respect to such measures

  $\Rightarrow$ We obtain counting results for both points and patches

Ratner-Marklof-Strömbergsson Measures [MS14]

- Fix $d+m=n$, $\mathbb{R}^n = V_{phys} \oplus V_{int}$, $W \subset V_{int}$. Define an embedding

$$ASL_d(\mathbb{R}) \hookrightarrow ASL_n(\mathbb{R}) \qquad (g,v) \mapsto \widetilde{(g,v)} = \left(\begin{pmatrix} g & 0_{d,m} \\ 0_{m,d} & I_m \end{pmatrix}, \begin{pmatrix} v \\ 0_m \end{pmatrix} \right)$$

- Let $\mathcal{L} \in Y_n = ASL_n(\mathbb{R}) / ASL_n(\mathbb{Z}) =$ space of grids, then

$$\underbrace{(g,v).\,\Lambda(\mathcal{L},W)}_{\text{orbit of a cut-and-project set}} = \Lambda(\underbrace{\widetilde{(g,v)}.\,\mathcal{L}}_{\text{orbit in the space of grids } Y_n}, W)$$

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Ratner Orbit closures $\overline{ASL_d(\mathbb{R})\,\mathcal{L}} \subset Y_n$ support $ASL_d(\mathbb{R})$-invariant probability measures described using Haar measures on algebraic groups

$$ASL_d(\mathbb{R}) < H < ASL_n(\mathbb{R}), \quad \overline{ASL_d(\mathbb{R})\,\mathcal{L}} = H\mathcal{L}$$

- Let $\bar{\mu}$ be an $ASL_d(\mathbb{R})$-invariant ergodic measure on Y_n, and

$\Psi(\underset{\text{grid}}{\mathcal{L}}) = \underset{\text{cut-and-project set in } \mathbb{R}^d}{\Lambda(\mathcal{L},W)}$. Then $\mu := \Psi_* \bar{\mu}$ RMS measure on $\{\Lambda(h\mathcal{L},W) \mid h \in H\}$

Classification of Measures

Theorem Any $ASL_d(\mathbb{R})$-invariant ergodic measure assigning full measure to irreducible cut-and-project sets is an RMS measure

Classification of Measures

Theorem Any $ASL_d(\mathbb{R})$-invariant ergodic measure assigning full measure to irreducible cut-and-project sets is an RMS measure

Theorem For every RMS measure, there exist $d \leq k \leq n$ and a real number field K so that $H = H' \ltimes \mathbb{R}^n$, $H' \subset SL_n(\mathbb{R})$. H' arises via restriction of scalars from the field K and one of the following groups:

- SL_k, then $n = k \cdot \deg(K/\mathbb{Q})$. Over $\mathbb{R}$ (and up to conjugation)

$$H' = \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{\deg(K/\mathbb{Q})} \end{pmatrix} \middle| A_j \in SL_k(\mathbb{R}) \right\}$$

- Sp_{2k}, then $n = 2k \cdot \deg(K/\mathbb{Q})$ (arises only when $d=2$)

Special Cases and Examples

- $\dim V_{phys} > \dim V_{int}$ or n prime $\Rightarrow H = ASL_n(\mathbb{R})$ ("generic case")

$\boxed{\begin{matrix} d \leq k \leq n \\ k \mid n \end{matrix}}$

- $\dim V_{phys} = \dim V_{int} = 2 \Rightarrow$ Three options
    - The generic case $(H = ASL_4(\mathbb{R}))$
    - $H = Sp_4(\mathbb{R}) \ltimes \mathbb{R}^4$ (can only arise if $d=2$)
    - $H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in SL_2(\mathbb{R}) \right\} \ltimes \mathbb{R}^4$, corresponding to restriction of scalars for SL_2 and $K = \mathbb{Q}(\sqrt{d})$.

Ammann-Beenker $K = \mathbb{Q}(\sqrt{2})$:

$$K = \mathbb{Q}(\sqrt{2}) \ni a + b\sqrt{d} \mapsto \begin{pmatrix} a & db \\ b & a \end{pmatrix} \in M_{\deg(K/\mathbb{Q})}(\mathbb{Q}) = M_2(\mathbb{Q})$$

$\Rightarrow$ extended entry-wise to define

$$\operatorname{Res}_{K|\mathbb{Q}}(SL_k) \cong \underbrace{(SL_k)^{\deg(K/\mathbb{Q})}}_{\text{over } \mathbb{R}} = (SL_2)^2$$

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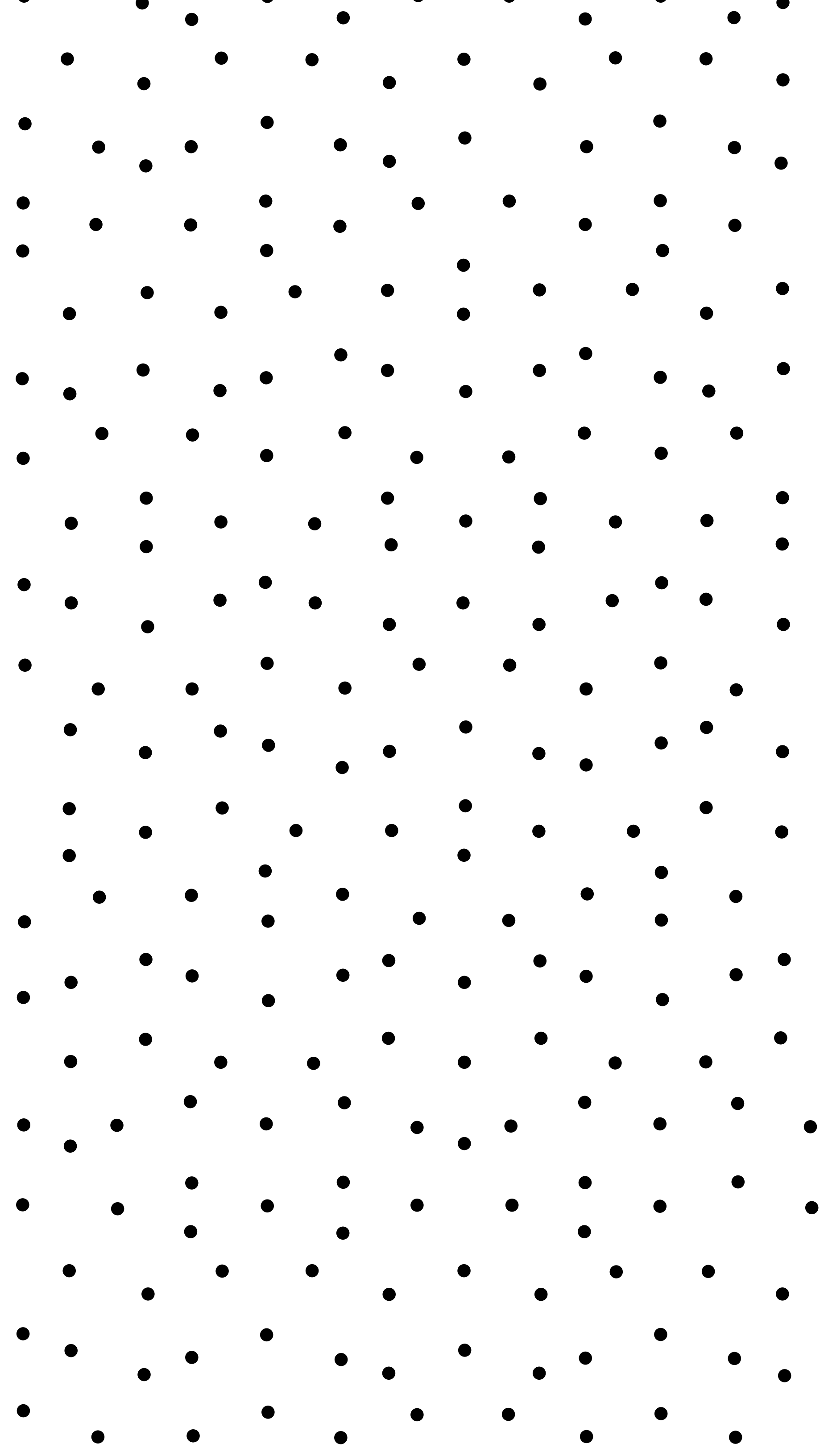
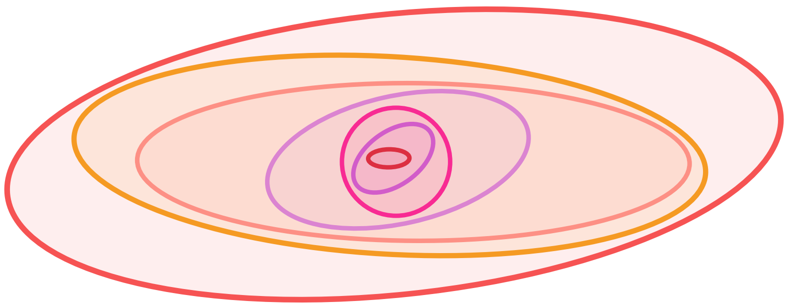


Effective Point Counting Following Schmidt

An unbounded ordered family is a collection of Borel subsets $\{\Omega_T \mid T \in \mathbb{R}_+\}$ of $\mathbb{R}^d$ so that

- $0 \leq T_1 \leq T_2 \Rightarrow \Omega_{T_1} \subset \Omega_{T_2}$
- For all T $\mathrm{vol}(\Omega_T) < \infty$
- $\mathrm{vol}(\Omega_T) \to \infty$



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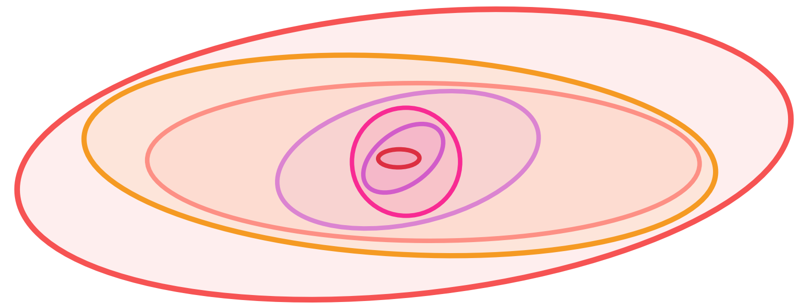


Theorem Let μ be an RMS measure. For every $\varepsilon > 0$, every unbounded ordered family and for μ-a.e. cut-and-project set Λ

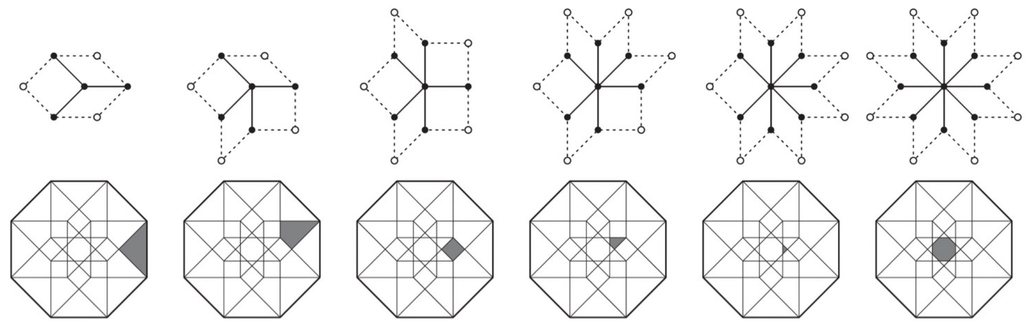
$$\#(\Lambda \cap \Omega_T) = D(\Lambda)\,\mathrm{vol}(\Omega_T) + O\left(\mathrm{vol}(\Omega_T)^{\frac{1}{2}+\varepsilon}\right)$$

matches best known result even for lattices and $\Omega_T = B(0,T)$

Effective Patch Counting Following Schmidt

For a point $x \in \Lambda$ and $R > 0$, the R-patch of Λ at x is

$$P_{\Lambda,R}(x) := (\Lambda - x) \cap B(0,R)$$



From Baake and Grimm's Aperiodic Order Vol 1

Theorem Let μ be an RMS measure and assume the window W has $\dim_B \partial W < m = \dim V_{int}$. There is $\theta > 0$ so that for any unbounded ordered family, for μ-a.e Λ and any patch P in Λ

$$\#\{x \in \Lambda \cap \Omega_T \mid P_{\Lambda,R}(x) = P\} = D(\Lambda,P)\operatorname{vol}(\Omega_T) + O(\operatorname{vol}(\Omega_T)^{1-\theta})$$

For $\dim_B \partial W = m-1$ any $\theta < \frac{1}{m+2}$ is good

A Siegel Summation Formula and a Rogers Second Moment Bound

Let $f \in C_c(\mathbb{R}^d)$ and μ an RMS measure. Define a

Siegel-Veech transform $\hat{f}(\Lambda) := \sum_{x \in \Lambda} f(x)$

Marklof-Strömbergsson There exists $c > 0$ so that

$$\int \hat{f}(\Lambda)\, d\mu(\Lambda) = c \int_{\mathbb{R}^d} f(x)\, d\mathrm{vol}(x)$$

← Siegel summation formula

Theorem There exists $C > 0$ so that if in addition

$f : \mathbb{R}^d \to [0,1]$ and $\hat{f} \in L^2(\mu)$, then

$$\int \left| \hat{f}(\Lambda) - \int \hat{f}(\Lambda)\, d\mu(\Lambda) \right|^2 d\mu(\Lambda) \leq C \int_{\mathbb{R}^d} f(x)\, d\mathrm{vol}(x)$$

← Rogers second moment bound

$\Rightarrow$ For counting: use $\hat{\mathbb{1}}_S(\Lambda) = \#\{\Lambda \cap S\}$

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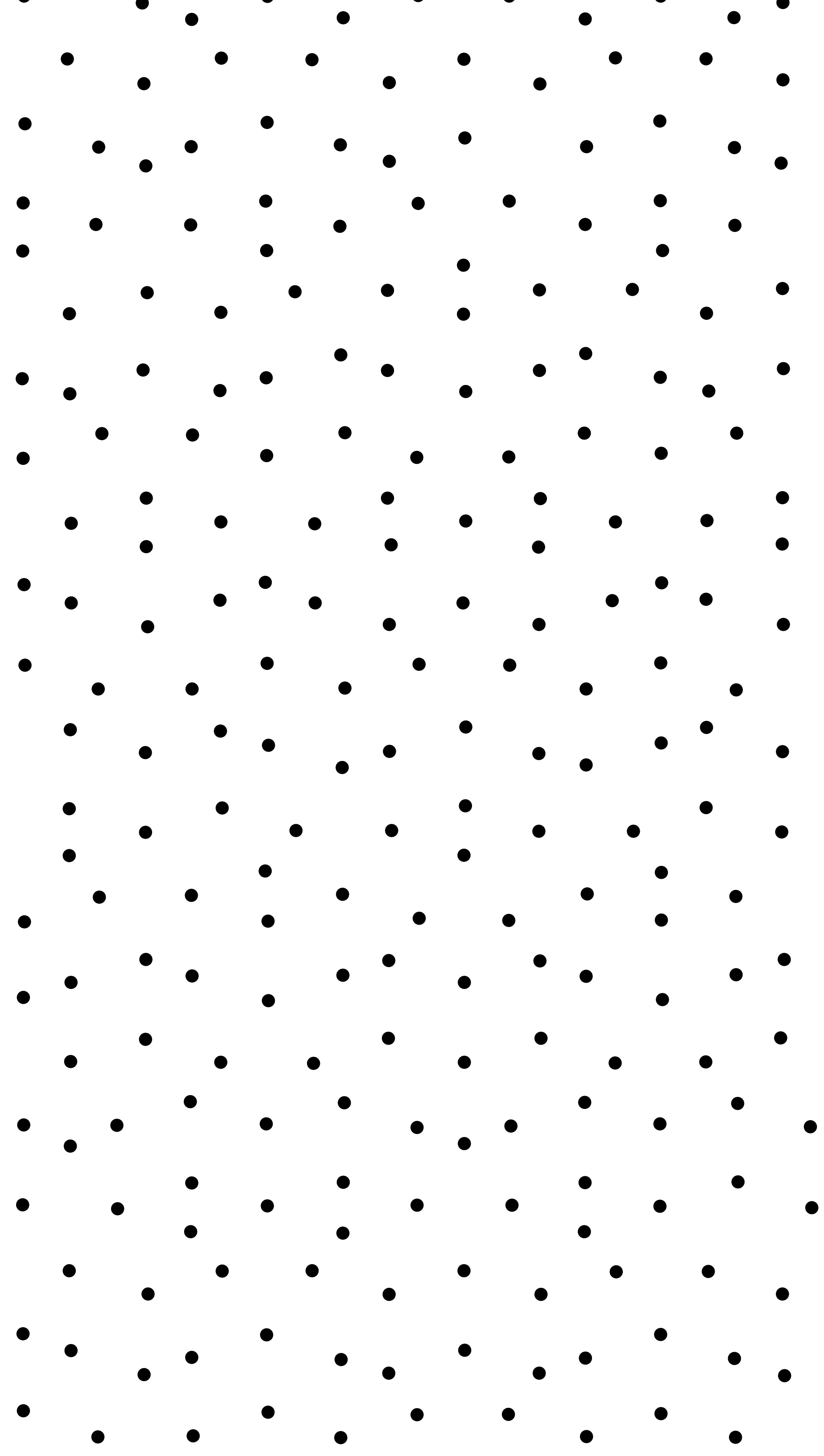

Thank You!